International Journal of Mathematics and
Computer Applications Research (IJMCAR)
ISSN(P): 2249-6955; ISSN(E): 2249-8060
Vol. 4, Issue 3, Jun 2014, 1-8

GENERALISED RETARDED INTEGRAL INEQUALITIES IN ONE VARIABLE

JAYASHREE PATIL

Department of Mathematics, Vasantrao Naik Mahavidyalaya, Aurangabad, Maharashtra, India

ABSTRACT

In this paper, we obtain some linear as well as nonlinear retarded integral inequalities which can be used also tools in certain applications.

KEYWORDS: Retarded Integral Inequalities, Explicit Bound, Boundedness

1. INTRODUCTION

Integral inequalities play an important role in the qualitative analysis of differential and integral equations. Many retarded inequalities have been discovered [1-12].Very recently Rashid in [12], obtained some new integral inequalities in one variables. In this paper we establish some new retarded integral inequalities, which generalize the main results of [12] which can be used as tools in the theory of differential equation with time delays which provide explicit bounds on unknown functions.

2. MAIN RESULTS

In what follows, R denotes the set of real numbers, $R_+ = [0, \infty)$, $R_+^* = (0, \infty)$, J = [a, b] are given subsets of R. Let $\Delta = J \times J$. C (J, R_+) denotes the set of all continuous functions from J into R_+ and C^1 (J, J) denotes the set of all continuous differentiable function from J into J.

Theorem 1: Let $u \in C(J, R_+)$, g, h, δ_t g, δ_t h $C \in (\Delta, R_+)$ and $f \in C(J, R_+)$, $\alpha \in C^1(J, J)$ be non-decreasing with $\alpha(t) \leq t$ on J. If the inequality

$$u(t) \leq f(t) + \int_a^t g(t, s)\, u(s)\, ds + \int_a^{\alpha(t)} h(t, s)\, u(s)\, ds \tag{2.1}$$

holds, then

$$u(t) \leq f(t) + \exp[G(t) + H(t)] \tag{2.2}$$

where

$$G(t) = \int_a^t g(t, s)\, ds \tag{2.3}$$

$$H(t) = \int_a^{\alpha(t)} h(t, s)\, ds \tag{2.4}$$

Proof: Since f (t) is positive and non-decreasing (2.1) can rewrite as

$$\frac{u(t)}{f(t)} \le 1 + \int_a^t g\,(t,s)\,f(s)\,ds + \int_a^{\alpha(t)} h\,(t,s)\,f(s)\,ds$$

Let $r(t) = \frac{u(t)}{f(t)}$ then

$$r(t) \le 1 + \int_a^t g\,(t,s)\,r(s)\,ds + \int_a^{\alpha(t)} h\,(t,s)\,r(s)\,ds \tag{2.5}$$

Define a function z (t) by the right-hand side of (2.5) then we have

$$z(t) = 1 + \int_a^t g\,(t,s)\,r(s)\,ds + \int_a^{\alpha(t)} h\,(t,s)\,r(s)\,ds \tag{2.6}$$

and $r(t) \le z(t)$, $z(a) = 1$ (2.7)

Differentiating (2.6) with respect to t, we get

$$z'(t) = g(t,t)\,r(t) + \int_a^{(t)} \partial_t\, g(t,s)\,r(s)\,ds + h(t,\alpha(t))\,r(\alpha(t)\,\alpha'(t) + \int_a^{\alpha(t)} \partial_t\, h(t,s)\,r(s)\,ds \tag{2.8}$$

using (2.7) we have,

$$z'(t) \le g(t,t)\,z(t) + \int_a^t \partial_t\, g(t,s)\,z(s)\,ds + h(t,\alpha(t))\,z(\alpha(t)\,\alpha'(t) + \int_a^{\alpha(t)} \partial_t\, h(t,s)\,r(s)\,ds$$

$$z'(t) \le z(t)\,\{g(t,t) + \int_a^t \partial_t\, g(t,s)\,ds + h(t,\alpha(t))\,\alpha'(t) + \int_\sigma^{\alpha(t)} \partial_t\, h(t,s)\,ds\}$$

$$\frac{z'(t)}{z(t)} \le \frac{d}{dt}\left(\int_a^t g(t,s)\,ds\right) + \frac{d}{dt}\left(\int_a^{\alpha(t)} h(t,s)\,ds\right)$$

Integrating above inequality from a to t, we get (2.9)

$$z(t) \le \exp\left[\int_a^t g(t,s)\,ds + \int_a^t h(t,s)\,ds\right]$$

So

$z(t) \le \exp[G(t) + H(t)]$

where G (t) and H (t) are defined by (2.3) and (2.4)

As $r(t) \le z(t)$, we get

$r(t) \le \exp[G(t) + H(t)]$

Hence

$u(t) \leq f(t) \exp[G(t) + H(t)]$.

Theorem 2: Let $u \in C(J, R_+)$, g, h, $\delta_t g$, $\delta_t h \in C(\Delta, R_+)$ and $F \in C(J, R_+)$, $\alpha \in C^1(J, J)$ be non-decreasing with $\alpha(t) \leq t$ on J and $p > 1$ is a constant. If the inequality

$$u^P(t) \leq f^P(t) + \int_a^t g(t,s)\, u(s)\, ds + \int_a^{\alpha(t)} h(t,s)\, u(s)\, ds \tag{2.10}$$

holds, then

$$u(t) \leq f(t)\left[1 + \left(\frac{p-1}{p}\right)[Q(t) + W(t)]\right]^{\frac{1}{p-1}} \tag{2.11}$$

where

$$Q(t) = \int_a^t f^{1-P}(s)\, g(t,s)\, ds \tag{2.12}$$

and

$$W(t) = \int_a^{\alpha(t)} f^{1-P}(s)\, h(t,s)\, ds \tag{2.13}$$

Proof: Since f (t) is positive and non-decreasing we can rewrite (2.10) as

$$\frac{u^p(t)}{f^p(t)} \leq 1 + \int_a^t g(t,s)\, f^{1-P}(s) \frac{u(s)}{f(s)}\, ds + \int_a^{\alpha(t)} (t,s)\, f^{1-P}(s) \frac{u(s)}{f(s)}\, ds \tag{2.14}$$

Let $r(t) = \dfrac{u(t)}{f(t)}$, then

$$r^P(t) \leq 1 + \int_a^t g(t,s)\, f^{1-P}(s)\, r(s)\, ds + \int_a^{\alpha(t)} (t,s)\, f^{1-P}(s)\, r(s)\, ds \tag{2.15}$$

Define a function z (t) by the right-hand side of (2.15) then we have

$$z(t) = 1 + \int_a^t g(t,s)\, f^{1-P}(s)\, r(s)\, ds + \int_a^{\alpha(t)} h(t,s)\, f^{1-P}(s)\, r(s)\, ds$$

Then it is clear that (2.16)

$$r^P(t) \leq z(t),\ z(a) = 1 \tag{2.17}$$

Differentiating (2.16) with respect to t, we get

$$z'(t) = g(t,t)\, f^{1-P}(t)\, r(t) + \int_a^t \partial_t\, g(t,s)\, f^{1-P}(s)\, r(s)\, ds + h(\alpha(t),t)\, f^{1-P}(\alpha(t))\, r(\alpha(t))$$

$$\alpha'(t) + \int_a^{\alpha(t)} \partial_t\, h(t,s,)\, f^{1-P}(s)\, r(s)\, ds \tag{2.18}$$

Using (2.17) in (2.18), we get,

$$z'(t) \le g(t,t)\, f^{1-P}(t)\, z^{\frac{1}{P}}(t) + \int_a^t \partial_t\, g(t,s)\, f^{1-P}(s)\, z^{\frac{1}{P}}(s) + h(t, \alpha(t))\, f^{1-P}(\alpha(t))\, z^{\frac{1}{P}}(\alpha(t))\, \alpha'(t) + \int_a^{\alpha(t)} \partial_t\, h(t,s)\, f^{1-P}(s)\, z^{\frac{1}{P}}(s)\, ds$$

Hence,

$$z'(t).\, z^{-\frac{1}{P}}(t) \le g(t,t)\, f^{1-P}(t) + \int_a^t \partial_t\, g(t,s)\, f^{1-P}(s)\, ds + h(t, \alpha(t))\, f^{1-P}(\alpha(t))\, \alpha'(t) + \int_a^{\alpha(t)} \partial_t\, h(t,s)\, f^{1-P}(s)\, ds \tag{2.19}$$

or

$$\frac{dz(t)}{z^{\frac{1}{P}}(t)} \le \frac{d}{dt}\left(\int_a^t g(t,s) f^{1-P}(s) ds\right) + \frac{d}{dt}\left(\int_a^{\alpha(t)} h(t,s)\, f^{1-P}(s) ds\right) \tag{2.20}$$

Integrating from a to t and making change of variable, we have

$$\frac{P}{P-1}\, z^{\frac{P-1}{1}} - \frac{P}{P-1} \le \int_a^t g\,(t,s)\, f^{1-P}(s)\, ds + \int_a^{\alpha(t)} h\,(t,s)\, f^{1-P}(s)\, ds + c \tag{2.21}$$

Using z (a) = 1, we have $c \ge 0$, Hence

$$z(t) \le \left[1 + \left(\frac{P-1}{P}\right)\left[\int_a^t g(t,s) f^{1-P}(s) ds + \int_a^t h(t,s) f^{1-P}(s) ds\right]\right]^{\frac{P}{P-1}}$$

OR

$$z(t) \le \left[1 + \left(\frac{P-1}{P}\right)\left[Q(t) + W(t)\right]\right]^{\frac{P}{P-1}} \tag{2.22}$$

Where Q (t) and W (t) are defined by (2.12) and (2.13).

Using (2.17) in (2.22) we get,

$$r(t) \le \left[1 + \left(\frac{P-1}{P}\right)\left[Q(t) + W(t)\right]\right]^{\frac{1}{P-1}}$$

so

$$u\ (t) \le f\ (t)\ \left[1+\left(\frac{P-1}{P}\right)[Q(t)+W(t)]\right]^{\frac{1}{P-1}}.$$

Theorem 2.3: Let u ∈ C (J, R_+), g, h, δ_t g, δ_t h ∈ C (Δ, R_+) and f ∈ C (J, R_+), α ∈ C^1 (J, J) be non-decreasing with α (t) ≤ t on J. For i = 1,2, let ψ_i C (R_+, R_+) be non-decreasing function with ψ_i (u) > 0 for u > 0 and $\frac{\psi_i u(t)}{f(t)} < \psi_i\left(\frac{u(t)}{f(t)}\right)$.

If the inequality

$$u\ (t) \le f\ (t) + \int_a^t g\ (t,s)\ \psi_1\ (u\ (s))\ ds + \int_a^t h\ (t,s)\ \psi_2\ (u(s))\ ds \tag{2.23}$$

Then for a ≤ t ≤ t,

i) in case ψ_1 (u) ≤ ψ_1 (u)

$$u\ (t) \le f\ (t)\ \phi_2^{-1}\ [\phi_2\ (1) + G\ (t) + H\ (t)] \tag{2.24}$$

ii) in case ψ_2 (u) ≤ ψ_1 (u)

$$u\ (t) \le f\ (t)\ \phi_1^{-1}\ [\phi_1\ (1) + G\ (t) + H\ (t)] \tag{2.25}$$

where G (t) and H (t) are defined by (2.3) and (2.4) and for i = 1,2 ϕ_t^{-1} are the inverse functions of

$$\phi_i\ (r) = \int_{r_0}^{r} \frac{ds}{\psi_i(s)},\ r > 0,\ r_0 > 0$$

and t_1 ∈ J is chosen so that ϕ_1 (1) + G (t) + H (t) ∈ Dom (ϕ_i^{-1}), respectively, for all t in [a,t_1]

Proof: Since f (t) is positive and non-decreasing we can restate (2.23) as

$$\frac{u(t)}{f(t)} \le 1 + \int_a^t g\ (t,s)\ \frac{\psi_1(u(s))}{f(s)}\ ds + \int_a^{\alpha(t)} h\ (t,s)\ \frac{\psi_2(u(s))}{f(s)}\ ds$$

$$\le 1 + \int_a^t g\ (t,s)\ \psi_1\left(\frac{u(s)}{f(s)}\right) ds + \int_0^{\alpha(t)} h\ (t,s)\ \psi_2\left(\frac{u(s)}{f(s)}\right) ds$$

Let r (t) = $\frac{u(t)}{f(t)}$. Hence, we have

$$r\ (t) \le 1 + \int_a^t g\ (t,s)\ \psi_1\ (r\ (s))\ ds + \int_a^{\alpha(t)} h\ (t,s)\ \psi_2\ (r\ (s))\ ds \tag{2.26}$$

Define z (t) by the right-hand side of (2.26), we have

$$z\ (t) = 1 + \int_a^t g\ (t,s)\ \psi_1\ (r\ (s))\ ds + \int_a^{\alpha(t)} h\ (t,s)\ \psi_2\ (r\ (s))\ ds \tag{2.27}$$

Then it is clear that

$$r(t) \le z(t),\ z(a) = 1 \tag{2.28}$$

Now,

$$z'(t) = g(t,t)\ \psi_1(r(t)) + \int_a^t \partial_t\ g(t,s)\ \psi_1(r(s))\,ds + h(t, \alpha(t))\ \alpha'(t)$$

$$+ \int_a^{\alpha(t)} \partial_t\ h(t,s)\ \psi_2(r(s))\,ds$$

$$\le g(t,t)\ \psi_1(z(t)) + \int_a^t \partial_t\ g(t,s)\ \psi_1(z(s))\,ds + h(t, \alpha(t))\ \psi_2(z(t))\ \alpha'(t)$$

$$+ \int_a^{\alpha(t)} \partial_t\ h(t,s)\ \psi_2(z(s))\,ds$$

In case $\psi_1(r(t)) \le \psi_2(r(t))$, we have

$$z'(t) \le \psi_2(z(t))\ [g(t,t) + \int_a^t \partial_t\ g(t,s)\,ds + h(t,\alpha,(t))\ \alpha'(t) + \int_a^t \partial_t\ h(t,s)\,ds]$$

$$z'(t) \le \psi_2(z(t)) \left[\frac{d}{dt}\left(\int_a^t g(t,s)ds\right) + \frac{d}{dt}\left(\int_a^{\alpha(t)} h(t,s)ds\right)\right]$$

there fore

$$\frac{d}{dt}\phi_2(z(t)) = \frac{z^1(t)}{\psi_2(z(t))} = \frac{d}{dt}\left(\int_a^t g(t,s)ds\right) + \frac{d}{dt}\left(\int_a^{\alpha(t)} h(t,s)ds\right) \tag{2.29}$$

Integrating (2.29) from a to t and using condition z (a) = 1, we get

$$\phi_2(z(t)) = \int_a^t g(t,s)\,ds + \int_a^{\alpha(t)} h(t,s)\,ds + \phi_2(1) \tag{2.30}$$

Hence

$$z(t) = \phi_2^{-1}[\int_a^t g(t,s)\,ds + \int_a^{\alpha(t)} h(t,s)\,ds + \phi_2(1)] \tag{2.31}$$

$$\text{i.e. } z(t) = \phi_2^{-1}[G(t) + H(t) + \phi_2(1)] \tag{2.32}$$

Using (2.28) in (2.32), we get the desired result. Since the proof of case (ii) is similar to case (i) we omit the details.

3. APPLICATIONS

In this section we present application of the inequality in Theorem 1 to study the boundedness of the solutions of the retarded differential equations.

First we consider the functional differential equation

$$x'(t) = F\,(s,t,\, x(t),\, x(t - h\,(t)) \tag{3.1}$$

with initial condition

$$x(c) = x_0,\; x \geq 0 \tag{3.2}$$

where $F \in C\,(J \times J \times R^2, R)$, $h \in C^1\,(J, R_+)$ such that $t - h\,(t) \geq 0$, $h'\,(t) < 1$ and $h\,(0) = 0$.

The following theorem deals with a bound on the solution of the problem (3.1).

Theorem 3.1: Assume that $F: J \times J \times R^2 \to R$ is a continuous function. There exists continuous function g (s,t), h (s,t) for s, t, $\in$ J such that

$$|F\,(s,t,u,v)| \leq g\,(s,t)\,|u| + h\,(s,t)\,|v| \tag{3.3}$$

and $|x_0| \leq k$ where $k > 0$ is a constant and let $M = \max\limits_{t \in J} \dfrac{1}{1 - h'(t)}$ (3.4)

If x (t) is any solution of (3.1) then

$$|x(t)| < |x_0| \exp\left(\int_a^t g(s,t)dt + \int_a^{\alpha(t)} \bar{h}(s,\tau)d\tau\right)$$

Proof: The solution x (t) of the problem (3.1) can be written as

$$x(t) = x_0 + \int_a^t F\,(s,\, \sigma,\, x\,(\sigma),\, x\,(\sigma - h\,(\sigma)))\, d\sigma \tag{3.4}$$

using (3.2), (3.3), (3.4) and making change of variables, we have

$$|x(t)| \leq |x_0| + \int_a^t g\,(s,\, \sigma)\, |x\,(\sigma)|\, d\sigma + \int_a^t h\,(s,\, \sigma)\, (x\,(\sigma - h\,(\sigma)))\, d\sigma \tag{3.5}$$

$$\leq |x_0| + \int_a^t g\,(s,\, \sigma)\, |x\,(\sigma)|\, d\sigma + \int_a^{t-h(t)} \bar{h}\,(\tau)\, |x\,(\tau)|\, d\tau \tag{3.6}$$

for $t \in J$ where $\bar{h}\,(\tau) = M\, h\,(\tau + h\,(\sigma)$, $\sigma, \tau \in J$.

Now a suitable application of the inequality in Theorem 2.1 to (3.6) yields the results.

CONCLUSIONS

The above inequalities can be extended to two variable case, which can further generalized to study qualitative properties of partial differential equations.

REFERENCES

1. Y. G. Sun, On retarded integral inequalities and their applications, J. Math, Anal. Appl.301 (2005) 265-275.+
2. A Morro, A Gronwall-like inequality and its applications to continuum thermodynamics, Boll, Uni. Mat. Ital. B6 (1982) 553-562.
3. B.G. Pachpatte, on a certain retarded integral inequality and its applications, J. Inequal. Pure, Appl. Math 5 (2004) (Article 19).
4. Run Xu, et al., on retarded integral inequalities in two independent variables and their applications, Appl. Math. Comput. 182 (2006) 1260-1266.
5. O. Lipovan, Integral inequalities for retarded volterra equations, J. Math. Anal. Appl. 322 (2006) 349-358.
6. O. Lipovan, A retarded Gronwall-like inequality and its applications, J.Math.Anal. Appl 252 (2000) 389-401.
7. Fanewi Meng, et al., on some new integral inequalities and their applications, Appl. Math. Comp. 148 (2004) 381-392.
8. O. Lipovan, A retarded integral inequality and its applications, J. Math Anal. Appl. 285 (2003) 436-443.
9. T. H. Gronwall, Note on the derivatives with respect to a parameter of the solutions of a system of differential equation, Ann. Math. 20 (1919) 292-296.
10. B. G. Pachpatte, Explicit bounds on a certain integral inequalities, J. Math. Anal. Appl. 267 (2002) 48-61.
11. B. G. Pachpatte, on certain retarded integral inequality and applications, J. Inequal. Pure. Appl. Maths. 3 (2004) (Article 18).
12. M. H. M. Rashid, Explicit bounds on retarded Gronwall-Bellman inequality, Tamk. J. Math 43 (2012) 99-108.